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Galilei-type transformations for integrable evolution equations

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Abstract. We consider integrable hierarchies of evolution equations defined with the help of the hereditary recursion operator which is related to the auxiliary second-order linear eigenvalue problem with energy-dependent potential. Explicit formulae of symmetry transformations generated by the shift of the spectral parameter, λ, are derived. We refer to such transformations as Galilei-type ones this because particular case is the well known Galilei transformation for the Korteweg–de Vries (KdV) equation. We apply a symmetry method for simultaneous construction of the invariant solution of the first two members of the KdV hierarchy.

1. Introduction and preliminaries

Symmetries of partial differential equations are used for the description of the general set of solutions, for description of conservation laws, for producing families of solutions from known exact solutions, etc [7].

The aim of this paper is to suggest the construction of symmetry transformations of Galilei-type for some class of integrable evolution equations which is isospectral relative to the underlying second-order eigenvalue problem with energy-dependent potential. In the framework of the inverse scattering transformation method, symmetry transformations are generated by the shift of the spectral parameter $\lambda \to \lambda - \tau$. The idea behind this approach is that the shift of the parameter λ will conserve the form of the auxiliary spectral problem.

Now we explain some relevant notions which are useful throughout this paper. Let M be a manifold of the smooth vector-functions $u : \mathbb{R} \to \mathbb{C}^n$. We denote by A_u, the algebra of polynomials in the finite collection of variables u^i_k, where the subscript k means the k-order derivative of some function $u^i = u^i(x)$ with respect to variable $x \in \mathbb{R}$.

Definition 1. *Let $X[u] = (X^1[u], \ldots, X^n[u])^T \in T_u M$ is a vector field and $\Lambda : T_u M \to T_u M$ is a linear operator. The Gateaux derivatives of X and Λ with respect to u in the direction $K \in T_u M$ are defined through the relations*

$$X'[u](K) = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} X[u + \varepsilon K] \qquad \Lambda'[u](K) = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} \Lambda[u + \varepsilon K].$$

Definition 2. *The Lie derivatives of X and Λ in the direction K are defined, respectively, as*

$$L_K X = X'(K) - K'(X) \qquad L_K \Lambda = \Lambda'(K) - [K', \Lambda].$$

The linear space $T_u M$ endowed with the commutator $[X, Y] = L_X Y$ bears the structure of the infinite Lie algebra Vect.

Definition 3. *The operator* $\Lambda : T_uM \to T_uM$ *is called hereditary if its Nijenhuis torsion vanishes [6], i.e.*

$$T_\Lambda(X, Y) = [\Lambda X, \Lambda Y] - \Lambda\{[\Lambda X, Y] + [X, \Lambda Y]\} + \Lambda^2[X, Y] = 0$$

for any $X, Y \in T_uM$.

It is known that vector fields $\Lambda^{k-1}X_1$ span Abelian subalgebra in Vect if an operator Λ is hereditary and $L_{X_1}\Lambda = 0$ [2]. In this case, one can define the hierarchy of commuting flows as

$$u_{t_k} = X_k[u] = \Lambda^{k-1}X_1. \tag{1}$$

In section 2 we investigate the hierarchies of the evolution equations of the form (1) with recursion operator, Λ, essentially connected with the auxilliary linear equation (2). Namely, invariant (with respect to some linear one-parameter point transformation) solutions to simultaneous sets of equations of integrable hierarchies are discussed.

2. Symmetry transformations generated by the shift of spectral parameter

Let us consider the stationary Schrödinger equation

$$\psi_{xx}(x, \lambda) + (u(x, \lambda) + (-\lambda)^n)\psi(x, \lambda) = 0 \tag{2}$$

with energy-dependent potential $u(x, \lambda) = \sum_{i=1}^n u^i(x)(-\lambda)^{i-1}$. It is known that, for every $n \in \mathbb{N}$, one can associate with equation (2) the hierarchy of completely integrable (isospectral) systems of evolution equations [4]:

$$u_{t_k} = X_k[u] = \Lambda^{k-1}[u]u_x \tag{3}$$

where $u = (u^1, \ldots, u^n)^T$, with the hereditary recursion operator of the form

$$\Lambda[u] = \begin{pmatrix} 0 & 0 & \ldots & 0 & \frac{1}{4}\partial_x^2 + u^1 + \frac{1}{2}u_x^1\partial_x^{-1} \\ -1 & 0 & \ldots & 0 & u^2 + \frac{1}{2}u_x^2\partial_x^{-1} \\ 0 & -1 & \ddots & \vdots & u^3 + \frac{1}{2}u_x^3\partial_x^{-1} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \ldots & 0 & -1 & u^n + \frac{1}{2}u_x^n\partial_x^{-1} \end{pmatrix}. \tag{4}$$

In equation (3), $X_1 = u_x$ and it is evident that $L_{X_1}\Lambda = 0$.

It is worth making a more precise definition of the operator ∂_x^{-1}. Let $A_u^0 \subset A_u$ be a ring of differential polynomials in the fields u^i with zero constant terms. We define ∂_x^{-1} by requiring that $\partial_x^{-1}(f) \in A_u^0$ for any $f = f[u] \in \operatorname{Im}\partial_x \subset A_u^0$. In particular case, if the functions $u^i = u^i(x)$ are in the Schwartz space $\mathcal{S}(\mathbb{R})$ then

$$\partial^{-1}(\cdot) \stackrel{\text{def}}{=} \int_{-\infty}^x \cdot \mathrm{d}x'.$$

Let us now define the one-parameter linear transformation of dependent variables $u = (u^1, \ldots, u^n)^T$

$$u = A(\tau)\bar{u} + d(\tau) \tag{5}$$

by the shift of the spectral parameter λ through the relation

$$\sum_{i=1}^n u^i(x)(-\lambda)^{i-1} + (-\lambda)^n = \sum_{i=1}^n \bar{u}^i(x)(-\lambda + \tau)^{i-1} + (-\lambda + \tau)^n. \tag{6}$$

It is easy to see that from (6) it follows that

$$\begin{aligned} u^1 &= F^1(\bar{u}, \tau) = \sum_{i=1}^{n} \bar{u}^i \tau^{i-1} + \tau^n \\ u^{i+1} &= F^{i+1}(\bar{u}, \tau) = \frac{1}{i!} \frac{\partial^i F^1}{\partial \tau^i} \qquad i = 1, \ldots, n-1. \end{aligned} \tag{7}$$

Indeed, if we substitute (7) into the left-hand side of (6) we obtain the Taylor series

$$\sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i F^1(\bar{u}, \tau)}{\partial \tau^i} (-\lambda)^i = F^1(\bar{u}, -\lambda + \tau).$$

Writing equation (7) explicitly we obtain transformation (5) in the form

$$u^i = \bar{u}^i + \sum_{k>i} A^i_k(\tau) \bar{u}^k + d^i(\tau)$$

with

$$A^i_k(\tau) = C^{i-1}_{k-1} \tau^{k-i} \qquad d^i(\tau) = C^{i-1}_n \tau^{n-i+1}$$

where $C^q_p \stackrel{\text{def}}{=} \frac{p!}{(p-q)!q!}$. From equation (6), it is obvious that the properties

$$A(-\tau) = A^{-1}(\tau) \qquad d(-\tau) = -A^{-1}(\tau) d(\tau)$$

are valid. So, the inverse transformation to (5) reads

$$\bar{u} = A(-\tau) u + d(-\tau).$$

The following lemma informs us about the transformation property of $\Lambda[u]$ with respect to (5).

Lemma. *The identity*

$$\Lambda[u]|_{u=F(\bar{u},\tau)} = A(\tau)(\Lambda[\bar{u}] + \tau) A^{-1}(\tau) \tag{8}$$

holds.

Proof. It is easy to check (8) by straightforward computation rewriting it, for convenience, as

$$\Lambda[u]|_{u=F(\bar{u},\tau)} A(\tau) = A(\tau)(\Lambda[\bar{u}] + \tau). \tag{9}$$

Element-wise, relation (9) is written as

$$(1n) : \left(\tfrac{1}{4}\partial_x^2 + u^1 + \tfrac{1}{2}u^1_x \partial_x^{-1}\right)|_{u^1=F^1(\bar{u},\tau)} = \left(\tfrac{1}{4}\partial_x^2 + \bar{u}^1 + \tfrac{1}{2}\bar{u}^1_x \partial_x^{-1}\right) + \sum_{i=2}^{n} C^0_{i-1} \tau^{i-1} \left(\bar{u}^i + \tfrac{1}{2}\bar{u}^i_x \partial_x^{-1}\right) + C^0_{n-1} \tau^{n-1}$$

$$(kn) : (-1) C^{k-2}_{n-1} \tau^{n-k+1} + \left(u^k + \tfrac{1}{2}u^k_x \partial_x^{-1}\right)|_{u^k=F^k(\bar{u},\tau)} = \left(\bar{u}^k + \tfrac{1}{2}\bar{u}^k_x \partial_x^{-1}\right) + \sum_{i=k+1}^{n} C^{k-1}_{i-1} \tau^{i-k} \left(\bar{u}^i + \tfrac{1}{2}\bar{u}^i_x \partial_x^{-1}\right) + C^{k-1}_{n-1} \tau^{n-k+1} \qquad k = 2, \ldots, n$$

$$(1l) : 0 = C^0_{l-1} \tau + (-1) C^0_l \tau \qquad l = 1, \ldots, n-1$$

$$(kl) : (-1) C^{k-2}_{l-1} \tau^{l-k+1} = C^{k-1}_{l-1} \tau^{l-k+1} + (-1) C^{k-1}_l \tau^{l-k+1} \qquad l \geqslant k+1 < n,$$

$$(kk) : (-1) C^{k-2}_{k-1} \tau = \tau + (-1) C^{k-1}_k \tau \qquad k = 1, \ldots, n-1$$

$$(k-1, k) : -1 = -1 \qquad k = 2, \ldots, n-1$$

$$(k, l) : 0 = 0, k \leqslant l-2$$

which is evidently valid. □

Taking into account (8) we derive evolution equations for the fields $\bar{u}^i$ in the form

$$\bar{u}_{t_k} = (\Lambda[\bar{u}] + \tau)^{k-1}\bar{u}_x = \sum_{r=0}^{k-1} C_{k-1}^r \tau^{k-r-1}\Lambda^r[\bar{u}]\bar{u}_x. \tag{10}$$

It should be noted that integral operator ∂_x^{-1}, which now acts on Im $\partial_x \subset A_{\bar{u}}^0$ is uniquely defined by the condition

$$F_*^{-1} \circ \partial_x^{-1}(f) \in A_u^0 \tag{11}$$

where the map $F_* : A_u \to A_{\bar{u}}$ and its inverse $F_*^{-1} : A_{\bar{u}} \to A_u$ are generated by (7). Now we can compute the vector fields $\Lambda^r[\bar{u}]\bar{u}_x$, keeping in mind condition (11), for every fixed $r \in \mathbb{N}$. From the explicit form of the recursion operator (4) it is obvious that $\Lambda^r[\bar{u}]\bar{u}_x$ are expressed in terms of the vector fields $X_k[\bar{u}]$ as

$$\Lambda^{r-1}[\bar{u}]\bar{u}_x = X_r[\bar{u}] + \sum_{m=1}^{r-1} a_{rm}\tau^{r-m}X_m[\bar{u}] \tag{12}$$

where the coefficients a_{rm} depend only on n. Substituting (12) into (10) we obtain evolution equations

$$\bar{u}_{t_k} = X_k[\bar{u}] + \sum_{l=1}^{k-1} b_{kl}\tau^{k-l}X_l[\bar{u}]$$

with some coefficients b_{kl} to be determined.

Let us take any $N \geqslant 2$ and consider the set of the first $(N-1)$ systems

$$\begin{aligned}
\bar{u}_{t_2} &= X_2[\bar{u}] + b_{21}\tau\bar{u}_x \\
\bar{u}_{t_3} &= X_3[\bar{u}] + b_{32}\tau X_2[\bar{u}] + b_{31}\tau^2\bar{u}_x \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\bar{u}_{t_N} &= X_N[\bar{u}] + b_{N,N-1}\tau X_{N-1}[\bar{u}] + \cdots + b_{N1}\tau^{N-1}\bar{u}_x.
\end{aligned} \tag{13}$$

For every $N \geqslant 2$ we complete (5) by adding linear transformation of independent variables $(x, t_2, \ldots, t_N)$

$$\begin{aligned}
\bar{x} &= x + b_{21}\tau t_2 + b_{31}\tau^2 t_3 + \cdots + b_{N1}\tau^{N-1}t_N \\
\bar{t}_2 &= t_2 + b_{32}\tau t_3 + b_{42}\tau^2 t_4 + \cdots + b_{N2}\tau^{N-2}t_N \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\bar{t}_N &= t_N.
\end{aligned} \tag{14}$$

Taking into account (13) and (14) we have

$$\begin{aligned}
&\bar{u}_{\bar{t}_2} = X_2[\bar{u}], \\
&\bar{u}_{\bar{t}_3} + b_{32}\tau\bar{u}_{\bar{t}_2} = X_3[\bar{u}] + b_{32}\tau X_2[\bar{u}], \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\bar{u}_{\bar{t}_N} + b_{N,N-1}\tau\bar{u}_{\bar{t}_{N-1}} + \cdots + b_{N2}\tau^{N-2}\bar{u}_{\bar{t}_2} = X_N[\bar{u}] + b_{N,N-1}\tau X_{N-1}[\bar{u}] + \cdots + b_{N2}\tau^{N-2}X_2[\bar{u}],
\end{aligned}$$

from which we easily obtain

$$\bar{u}_{\bar{t}_k} = X_k[\bar{u}] \qquad k = 2, \ldots, N.$$

Summarizing the above, we can state the following corollary.

Corollary. *The simultaneous set of equations*

$$\begin{aligned} u_{t_2} &= X_2[u] = \Lambda[u]\, u_x \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_{t_N} &= X_N[u] = \Lambda^{N-1}[u] u_x \end{aligned} \tag{15}$$

are invariant with respect to the linear one-parameter transformation (5), (14).

So, if we know some solution $u(x, t_2, \ldots, t_N)$ of (15) we can construct a one-parameter solution family

$$u(\tau; x, t_2, \ldots, t_N) = A(\tau) u(\bar{x}, \bar{t}_2, \ldots, \bar{t}_N) + d(\tau) \tag{16}$$

where the collection of variables $(\bar{x}, \bar{t}_2, \ldots, \bar{t}_N)$ is expressed via $(x, t_2, \ldots, t_N)$ by virtue of (14).

Example 1. *Let us take $N = 2$. In this case, in the following formula we need*

$$\partial_x^{-1}(\bar{u}_x^n) = \bar{u}^n + n\tau \tag{17}$$

which follows from condition (11) since

$$F_*^{-1}(\bar{u}^n + n\tau) = u^n \in A_u^0.$$

Using (17) we have

$$\Lambda[\bar{u}]\bar{u}_x = X_2[\bar{u}] + \frac{n}{2}\tau\bar{u}_x$$

and

$$\bar{u}_{t_2} = (\Lambda[\bar{u}] + \tau)\, \bar{u}_x = X_2[\bar{u}] + \left(\frac{n}{2} + 1\right) \tau \bar{u}_x.$$

Thus, the transformation of dependent variables (5) should be completed by transformation for variables (x, t_2), as follows:

$$\begin{aligned} \bar{x} &= x + \left(\frac{n}{2} + 1\right) \tau t_2 \\ \bar{t}_2 &= t_2. \end{aligned} \tag{18}$$

(i) For $n = 1$, (5), (18) become the well known Galilei transformation (see, for example, [5])

$$\begin{aligned} \bar{u} &= u - \tau \\ \bar{x} &= x + \tfrac{3}{2}\tau t_2 \\ \bar{t}_2 &= t_2 \end{aligned}$$

for the KdV equation

$$u_{t_2} = \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}uu_x. \tag{19}$$

(ii) In the case $n = 2$, we have the transformation:

$$\begin{aligned} \bar{u}^1 &= u^1 - u^2\tau + \tau^2 \\ \bar{u}^2 &= u^2 - 2\tau \\ \bar{x} &= x + 2\tau t_2 \\ \bar{t}_2 &= t_2 \end{aligned}$$

for the Kaup system [1,3]

$$\begin{aligned} u^1_{t_2} &= \tfrac{1}{4}u^2_{xxx} + u^1u^2_x + \tfrac{1}{2}u^2u^1_x \\ u^2_{t_2} &= -u^1_x + \tfrac{3}{2}u^2u^2_x. \end{aligned} \tag{20}$$

Example 2. *Let us take $N = 3$. In this case, we are in a position to search for the symmetry transformation for the pair of systems:*

$$u_{t_2} = X_2[u] = \Lambda[u]u_x$$
$$u_{t_3} = X_3[u] = \Lambda^2[u]u_x.$$

To compute $\Lambda[\bar{u}]X_2$, in the following formulae we need:

$$\partial_x^{-1}\left(\tfrac{1}{4}\bar{u}_{xxx} + \tfrac{3}{2}\bar{u}\bar{u}_x\right) = \tfrac{1}{4}\bar{u}_{xx} + \tfrac{3}{4}(\bar{u})^2 - \tfrac{3}{4}\tau^2 \tag{21}$$

for $n = 1$ and

$$\partial_x^{-1}\left(-\bar{u}_x^{n-1} + \tfrac{3}{2}\bar{u}^n\bar{u}_x^n\right) = -\bar{u}^{n-1} + \tfrac{3}{4}(\bar{u}^n)^2 + \left(\tfrac{1}{2}n(n-1) - \tfrac{3}{4}n^2\right)\tau^2 \tag{22}$$

for $n \geqslant 2$. Using (21) and (22) we obtain†

$$\begin{aligned}\Lambda^2[\bar{u}]\bar{u}_x &= \Lambda[\bar{u}]X_2 + \frac{n}{2}\tau\Lambda[\bar{u}]\bar{u}_x \\ &= X_3 + \left(\frac{1}{4}n(n-1) - \frac{3}{8}n^2\right)\tau^2\bar{u}_x + \frac{n}{2}\tau\left(X_2 + \frac{n}{2}\tau\bar{u}_x\right) \\ &= X_3 + \frac{n}{2}\tau X_2 + \left(\frac{1}{4}n(n-1) - \frac{1}{8}n^2\right)\tau^2\bar{u}_x\end{aligned}$$

and

$$\begin{aligned}\bar{u}_{t_3} &= (\Lambda[\bar{u}] + \tau)^2\,\bar{u}_x \\ &= \Lambda^2[\bar{u}]\bar{u}_x + 2\tau\Lambda[\bar{u}]\bar{u}_x + \tau^2\bar{u}_x \\ &= X_3 + \left(\frac{n}{2} + 2\right)\tau X_2 + \frac{1}{2}\left(\frac{n}{2} + 1\right)\left(\frac{n}{2} + 2\right)\tau^2\bar{u}_x.\end{aligned}$$

Then, the transformation of variables (x, t_2, t_3) (14) takes the form

$$\begin{aligned}\bar{x} &= x + \left(\frac{n}{2} + 1\right)\tau t_2 + \frac{1}{2}\left(\frac{n}{2} + 1\right)\left(\frac{n}{2} + 2\right)\tau^2 t_3 \\ \bar{t}_2 &= t_2 + \left(\frac{n}{2} + 2\right)\tau t_3 \\ \bar{t}_3 &= t_3.\end{aligned}$$

Next, we compute the coefficients b_{kl} in explicit form. First, we compute $b_{k,k-1}$. Using (10) and (12) we obtain:

$$\begin{aligned}\bar{u}_{t_k} &= \Lambda^{k-1}[\bar{u}]\bar{u}_x + (k-1)\tau\Lambda^{k-2}[\bar{u}]\bar{u}_x + \mathrm{O}(\tau^2) \\ &= \Lambda^{k-2}\left(X_2 + \frac{n}{2}\tau\bar{u}_x\right) + (k-1)\tau\Lambda^{k-2}\bar{u}_x + \mathrm{O}(\tau^2) \\ &= \Lambda^{k-2}X_2 + \left(\frac{n}{2} + k - 1\right)\tau\Lambda^{k-2}\bar{u}_x + \mathrm{O}(\tau^2) \\ &= X_k + \left(\frac{n}{2} + k - 1\right)\tau X_{k-1} + \mathrm{O}(\tau^2).\end{aligned} \tag{23}$$

In equation (23) we have taken into account that

$$\Lambda[\bar{u}]X_2 = X_3 + \mathrm{O}(\tau^2)$$

which follows from (21), (22). From (23) we obtain that $b_{k,k-1} = \left(\frac{n}{2} + k - 1\right)$.

† We write Λ, X_k instead of $\Lambda[\bar{u}]$, $X_k[\bar{u}]$ so no confusion can arise.

Now let us rewrite (14) in the vector form $\bar{t} = B_N(\tau)t$, where $t = (x, t_2, \ldots, t_N)^T$ and $B_N(\tau)$ is the upper diagonal matrix

$$B_N(\tau) = \begin{pmatrix} 1 & b_{21}\tau & \ldots & b_{N1}\tau^{N-1} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{N,N-1}\tau \\ 0 & \ldots & 0 & 1 \end{pmatrix}.$$

We must require the validity of the group property

$$B_N(\tau)B_N(\tau_1) = B_N(\tau + \tau_1)$$

which can be written element-wise as

$$\begin{aligned} b_{r+k,k}(\tau + \tau_1)^r &= b_{r+k,k}\left(\tau^r + \sum_{i=1}^{r-1} C_r^i \tau^i \tau_1^{r-i} + \tau_1^r\right) \\ &= b_{r+k,k}\tau^r + \sum_{i=1}^{r-1} b_{k+r,k+i} b_{k+i,k} \tau^i \tau_1^{r-i} + b_{r+k,k}\tau_1^r. \end{aligned} \tag{24}$$

From (24) one sees that the relation

$$C_r^i b_{r+k,k} = b_{r+k,k+i} b_{k+i,k} \tag{25}$$

must be fulfilled for each $i = 1, \ldots, r-1$. As can be checked, the solution of equation (25) is given by

$$b_{k+r,k} = \frac{1}{r!}\prod_{i=1}^{r} b_{k+i,k+i-1} = \frac{1}{r!}\prod_{i=1}^{r}\left(\frac{n}{2} + k + i - 1\right).$$

So, we found matrix $B_N(\tau)$ for any $N \geqslant 2$.

3. Invariant solutions of KdV hierarchy equations

In section 2 we presented Galilei-type symmetry transformations, which are applicable for all members of integrable hierarchies generated by the recursion operator (4). Let us apply a symmetry method for construction of the simultaneous solution $u(x, t_2, t_3)$ of the first two members of the KdV hierarchy, which is invariant with respect to the corresponding Galilei-type group.

Consider a one-parameter group

$$\bar{u} = u - \tau \qquad \bar{x} = x + \tfrac{3}{2}\tau t_2 + \tfrac{15}{8}\tau^2 t_3 \qquad \bar{t}_2 = t_2 + \tfrac{5}{2}\tau t_3 \qquad \bar{t}_3 = t_3 \tag{26}$$

with generator

$$v = -\frac{\partial}{\partial u} + \frac{5}{2}t_3\frac{\partial}{\partial t_2} + \frac{3}{2}t_2\frac{\partial}{\partial x}.$$

It is easy to compute the differential invariants of this group. Namely, we have

$$\xi = t_3 \qquad z = xt_3 - \tfrac{3}{10}t_2^2 \qquad w = t_3u^2 + \tfrac{4}{5}t_2u + \tfrac{8}{15}x. \tag{27}$$

Solving the quadratic equation

$$t_3u^2 + \tfrac{4}{5}t_2u + (\tfrac{8}{15}x - w) = 0$$

we obtain

$$u = -\frac{2t_2}{5t_3} \pm \frac{(t_3w - \frac{8}{15}z)^{1/2}}{t_3} \qquad w = w(z, \xi). \tag{28}$$

It is convenient to use the ansatz $v = v(z, \xi) = \pm(\xi w - \frac{8}{15}z)^{1/2}$.

Substituting (28) into the third-order KdV equation (19) we obtain that it reduces to the parameter-dependent first Painlevé equation (P_I)

$$v_{zz} = -\frac{1}{\xi^3}\left(3v^2 + \frac{8}{5}z + c(\xi)\right) \tag{29}$$

where $c(\xi)$ is an arbitrary function—'constant of integration'.

Let us now substitute (28) into the fifth-order KdV equation

$$u_{t_3} = \tfrac{1}{16}u_x^{(V)} + \tfrac{5}{8}uu_{xxx} + \tfrac{5}{4}u_x u_{xx} + \tfrac{15}{8}u^2 u_x. \tag{30}$$

Taking into account equation (29) we arruve at the linear (!) evolution equation

$$v_\xi = -\left(\frac{6z}{5\xi} + \frac{c(\xi)}{8\xi}\right)v_z + \frac{3}{5\xi}v.$$

Simple calculations show that the compatibility condition $v_{zz\xi} = v_{\xi zz}$ is equivalent to equation $\xi c'(\xi) = c(\xi)$, whose solution is a homogeneous linear function of ξ. We put $c = 8\mu\xi$, where μ is an arbitrary constant.

Thus we can state that the ansatz

$$u(x, t_2, t_3) = \frac{-2t_2 + 5v(z, \xi)}{5\xi} \qquad \xi = t_3 \qquad z = xt_3 - \frac{3}{10}t_2^2$$

gives simultaneous invariant solutions of equations (19) and (30) provided that the function $v = v(z, \xi)$ satisfies the pair of compatible equations:

$$\begin{aligned} v_{zz} &= -\frac{1}{\xi^3}\left(3v^2 + \frac{8}{5}z + 8\mu\xi\right) \\ v_\xi &= -\left(\frac{6z}{5\xi} + \mu\right)v_z + \frac{3}{5\xi}v. \end{aligned} \tag{31}$$

It can be checked that the solution of (31) is given by

$$v(z, \xi) = af(Z) \qquad Z = bz + c$$

where $f = f(Z)$ is a solution of the P_I equation in its standard form

$$f'' = 6f^2 + Z$$
$$a = -2\varepsilon^2\xi^{3/5} \qquad b = \varepsilon\xi^{-6/5} \qquad c = 4\mu\varepsilon^{-4}\xi^{-1/5} \qquad \text{where } \varepsilon = (\tfrac{4}{5})^{1/5}.$$

4. Conclusion

The results of this paper are explicit formulae presenting the symmetry transformations of integrable evolution equations constructed with the help of the recursion operator. These transformations are generated by the shift of the spectral parameter $\lambda \to \lambda - \tau$. We would like to note that these transformations cannot be applied for a single equation in hierarchy, excluding the case of the first nontrivial member in each hierarchy such as the KdV equation (19) or the Kaup system (20).

We believe that this approach can be applied to integrable hierarchies associated with auxiliary eigenvalue problems of order greater than two.

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